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Spontaneous magnetization in the disorder-dominated phase of the two-dimensional random-bond Ising model

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Abstract. The self-consistent approach to the two-dimensional Ising model with quenched random bonds is extended to the full lattice theory described by four real fermions. A calculation of the averaged spin–spin correlation function for large separation of the spins in the disorder-dominated phase indicates an exponential decay of this quantity and therefore a vanishing spontaneous magnetization. The corresponding correlation length is proportional to $1/\eta^2$, where η denotes the order parameter of the new phase introduced by Ziegler.

1. Introduction

The 2D Ising model [1], defined on a square lattice with quenched disorder in the ferromagnetic bonds, is defined via the Hamiltonian

$$H = -\frac{1}{k_B T}\sum_{\langle i,j\rangle} J(i,j) S_i S_j.$$

The bond strengths are ferromagnetic random variables $0 < J(i,j)$ and $\overline{J(i,j)} = J_0$; $\langle i,j\rangle$ is a pair of nearest-neighbour sites.

Whereas it is well confirmed that this system undergoes a phase transition to a ferromagnetic ordered state at a critical temperature lower than that of the pure system, the nature of the transition is not completely understood because of the non-applicability of the Harris criterion and also because of several conflicting analytical results [2–4]. The only exactly solvable model for a disordered ferromagnet is the McCoy–Wu model [5], where the disorder is essentially one-dimensional. For a model with isotropic disorder, the famous analysis by Dotsenko and Dotsenko [4] mapped the system onto the $N = 0$ Gross–Neveu model to get the averaged thermodynamic quantities. This is possible because of the free fermion representation of the pure model. The authors used the replica trick and a continuum field theory, which was analysed with the momentum-space renormalization-group (RG) technique. This model is asymptotically free in the infrared, the coupling g ($\simeq$ disorder strength) is marginal irrelevant, the phase diagram is not changed and the specific heat diverges at the critical point but more slowly than in the pure model ($\simeq \log\log(|T-T_c|^{-1})$ versus $\log(|T-T_c|^{-1})$). The averaged spin correlation function $\overline{\langle S_0 S_n\rangle}$ shows a dramatic change, namely, a slow decay: $\overline{\langle S_0 S_n\rangle} \simeq \exp(-\frac{1}{g}(\log\log n)^2)$. The disorder seems to enhance the correlations instead of destroying them; the critical exponent goes from $\frac{1}{4}$ to zero.

An alternative approach uses the bosonized version of the pure model, which makes the spin operators local [6–8]. The disorder is treated with the same method as in [4]. The results confirm the behaviour of the specific heat but predict only logarithmic corrections to the correlation exponent. An equivalent way is to take advantage of the conformal invariance of the pure model [9]. All these methods suffer from two difficulties: first, the replica trick is used and, second, they start from the continuum limit associated with the supposed critical point. Now there exists an alternative to the replica trick, through a supersymmetric version of the effective model [3, 10]. This is important because the replica trick is questionable in our case [11, 12]. In fact, [10] performs a RG analysis of this model and [3] calculates the saddle-point structure of the theory after a transformation to composite operators (Q-matrices). Both [3] and [10] show finite specific heat in the critical region. Whereas the RG treatment in [10] lacks the correct incorporation of the additive renormalization crucial to extensive quantities [3, 13] shows that in the critical region a new saddle point becomes stable and governs the thermodynamic behaviour of the effective model. This saddle point is accompanied by spontaneous symmetry breaking and a new phase between the ferro- and paramagnetic one. The corresponding order parameter stems from a regularization term needed for the boson integration in the supersymmetric theory [3, 10]. The non-vanishing of the order parameter can be proven rigorously [14, 15], but there exist two regularizations corresponding to different physical situations, one to the random-bond Ising model, the other to a system of polymer chains [12].

All investigations mentioned so far use a large-scale approximation where two of four lattice-fermion degrees of freedom are ignored. The RG approaches do need the continuum limit. The difficulty of the continuum limit is connected with the fact that the renormalization procedure and the bosonization are not interchangeable. This is the 'technical' reason for the discrepancy between the findings of [4] and those of [5–7] regarding the spin correlation function [16].

It seems worthwile, therefore, to extract as much information on magnetic correlations as possible without using the large-scale approximation. This is the aim of the present paper. Moreover, the technique used avoids both the replica trick and the supersymmetric theory, staying close to the original model by direct average of the Green function in the framework of the $1/N$ expansion [2]. The parameter N serves as a bookkeeping device to derive a self-consistent theory which contains the tadpole structure completely (section 2). The diagrammatic expansion builds a bridge between the Q-matrix approach and the RG calculations by comparing the classes of Feynman diagrams which are accounted for. The tadpoles are usually omitted in field-theoretical investigations because they can be treated by normal ordering. But this may be dangerous if one does not know the vacuum structure, i.e. the phase diagram. With the full 4-fermion lattice theory one can identify the diagrams which yield a second-order contribution in the coupling to an 'exceptional' mass term (see section 2), which turns out to be equivalent to the regularization term in [3]. Eventually, this term (the order parameter of the new phase) cannot be viewed as an artefact of the supersymmetric theory. But the main reason to consider the full theory is the possibility of calculating magnetic correlations which are the subject of the many computer simulations performed on the model [17–19]. These show very good agreement with the thermodynamic predictions of [3] (see also [2]; an experimental result for the specific heat is also available [20]).

The disorder average of the square of the spontaneous magnetization in the new phase is now calculated in section 3. This is done by determination of the averaged spin correlation function for large separation of the spins (section 3). This quantity is related to the averaged square of the extensive spontaneous magnetization, i.e. $\overline{\mathcal{M}^2} = \overline{(\sum_i \langle S_i \rangle)^2}$, which is translationally invariant before the disorder average, by means of a Griffith inequality [21] and the cluster property [21, 22] together with the hypothesis of self-averaging of $\mathcal{M}$. (The spin correlation function itself, whose first moment is calculated in this paper, does not need to be self-averaging [9]. But the exponential decay of magnetic correlations $\overline{\langle S_i S_j \rangle}$ is sufficient for the conclusion that $\mathcal{M}$ vanishes in a *fixed* sample in the thermodynamic limit.) Even if the cluster property is questionable [22], our result ($\mathcal{M} = 0$) is not affected by this. However, $\overline{\mathcal{M}^2}$ should not be confused with the Edwards–Anderson order parameter for spin glasses [22]: $\overline{\langle S_i \rangle^2}$, a local quantity, which becomes translationally invariant after averaging over disorder. Therefore, the present methods do not allow an estimate of this important quantity in the new phase. We use Gaussian disorder to simplify the calculations although a bounded distribution of the bond strength is necessary to keep all the bonds ferromagnetic. Nevertheless, the only relevant cummulant in the $N = \infty$ limit is the second one. In [2] it was shown that the higher cummulants appear only in higher-order terms in the $1/N$-expansion.

2. The self-consistency equation

The N-fold replicated partition function for a specific configuration of disorder is

$$Z_{DN} = \int \mathcal{D}\xi \exp H_D$$

with the Euclidean action (Hamiltonian) [1, 2]

$$\begin{aligned} H_D = \sum_{\alpha,\beta}^{N} \sum_{r} & \xi_1^\alpha(r)\xi_2^\alpha(r) + \xi_3^\alpha(r)\xi_4^\alpha(r) + \xi_1^\alpha(r)\xi_4^\alpha(r) \\ & + \xi_1^\alpha(r)\xi_3^\alpha(r) + \xi_3^\alpha(r)\xi_2^\alpha(r) + \xi_2^\alpha(r)\xi_4^\alpha(r) \\ & + t_0\xi_1^\alpha(r)\xi_2^\alpha(r+e_y) + t_0\xi_3^\alpha(r)\xi_4^\alpha(r+e_x) \\ & + \delta t_y^{\alpha\beta}(r)\xi_1^\alpha(r)\xi_2^\beta(r+e_y) + \delta t_x^{\alpha\beta}(r)\xi_3^\alpha(r)\xi_4^\beta(r+e_x). \end{aligned} \tag{1}$$

The ξ_i^α are real Grassmann fields defined at the points r of a two-dimensional square lattice $\Lambda = Z \times Z$, $e_{x,y}$ are the unit vectors in the two directions of the lattice. $t_0 = \tanh J_0/kT$ measures the average bond strength. The disorder variables $\delta t_{x,y}^{\alpha\beta}$ are statistically independent and Gaussian distributed with mean zero and variance ($i, j = x, y$)

$$\overline{\delta t_i^{\alpha\gamma}(r)\delta t_j^{\beta\delta}(r')} = \frac{g}{N}\delta_{r,r'}\delta_{i,j}[\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}]. \tag{2}$$

For a given configuration D of disorder, one expands the Green function for a fixed index α:

$$G_{i,j}^{\alpha,D}(r, r') = \langle \xi_i^\alpha(r)\xi_j^\alpha(r') \rangle_{A_D}. \tag{3}$$

With

$$G_{i,j}^{\alpha,0}(r, r') = \langle \xi_i^\alpha(r)\xi_j^\alpha(r') \rangle_{\delta t=0} \tag{4}$$

one gets

$$G^{\alpha,D}_{i,j}(\boldsymbol{r},\boldsymbol{r}') = G^{\alpha,0}_{i,j} + \left\langle \xi^{\alpha}_{i}(\boldsymbol{r}) \sum_{m=1}^{\infty} \sum_{\alpha_k,\beta_k}^{N} \sum_{\boldsymbol{r}_1 \ldots \boldsymbol{r}_m} \prod^{m} \left[\delta t_y^{\alpha_k,\beta_k}(\boldsymbol{r}_k) \xi_1^{\alpha_k}(\boldsymbol{r}_k) \xi_2^{\beta_k}(\boldsymbol{r}_k + \boldsymbol{e}_y) \right.\right.$$
$$\left.\left. + \delta t_x^{\alpha_k,\beta_k} \xi_3^{\alpha_k}(\boldsymbol{r}_k) \xi_4^{\beta_k}(\boldsymbol{r}_k + \boldsymbol{e}_x) \right] \xi_j^{\alpha}(\boldsymbol{r}') \right\rangle_{\delta t = 0} . \tag{5}$$

Applying the fermionic Wick theorem and averaging over the disorder fields to get the averaged Green function $G'(\boldsymbol{r},\boldsymbol{r}') = \overline{G^{\alpha}(\boldsymbol{r},\boldsymbol{r}')}$ leads to a closed loop of G^0 propagators with self-contacts. As was shown previously [2], the only terms contributing to order N^0 have the tadpole structure. There are essentially four of them (i.e. eight for both axes), depicted in figure 1 and figure 2. The left graph of figure 1, for example, corresponds to

$$\frac{g}{N} \sum_{\beta}^{N} \sum_{\boldsymbol{r}_1} G^{\alpha,0}_{i,1}(\boldsymbol{r},\boldsymbol{r}_1) G^{\beta,0}_{2,1}(\boldsymbol{r}_1 + \boldsymbol{e}_y, \boldsymbol{r}_1) G^{\alpha,0}_{2,j}(\boldsymbol{r}_1 + \boldsymbol{e}_y, \boldsymbol{r}'). \tag{6}$$

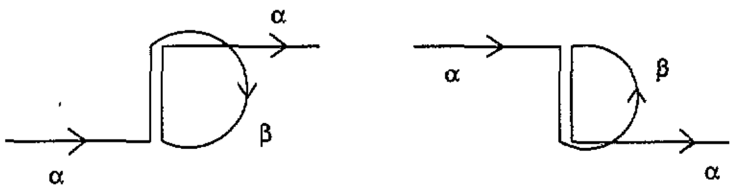

Figure 1. Two off-diagonal contributions to the propagator, coming from averaging over bond disorder in the y direction. The left graph corresponds to equation (6). α and β are indices of the N colors. α is fixed whereas β runs from 1 to N.

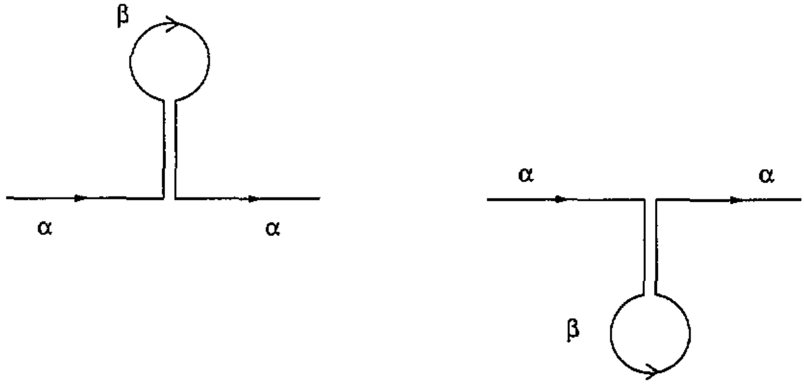

Figure 2. Two diagonal contributions. These vanish if $\eta = 0$.

Therefore one deduces a (nonlinear) self-consistency equation including all tadpoles. Dropping the replica indices it reads

$$G'_{k,l}(\boldsymbol{r},\boldsymbol{r}') = G^0_{k,l}(\boldsymbol{r},\boldsymbol{r}') + g\sum_{\boldsymbol{r}_1}\sum_{i}^{4}\sum_{j=i,\hat{i}}(-1)^{\delta_{i,j}}G^0_{k,i}(\boldsymbol{r},\boldsymbol{r}_1) \times G'_{\hat{i},j}(\boldsymbol{r}_1+e(\hat{i}),\boldsymbol{r}_1+e(j))G'_{\hat{j},l}(\boldsymbol{r}_1+e(\hat{j}),\boldsymbol{r}') \tag{7}$$

with

$$\hat{1}=2 \qquad \hat{2}=1 \qquad \hat{3}=4 \qquad \hat{4}=3$$

and

$$e(1) = -e(2) = -e_y$$
$$e(3) = -e(4) = -e_x.$$

The sign factor in (7) comes from an exchange of Grassmann variables in case of diagonal propagators in the loop (see figure 2). Equation (7) determines the 'self-energy' of the effective Hamiltonian. In momentum space, one gets, together with $H' = G'^{-1}$ and $H_0 = (G^0)^{-1}$,

$$H_0(\boldsymbol{p}) = H'(\boldsymbol{p}) + gC(\boldsymbol{p}). \tag{8}$$

C is a 4×4 matrix

$$C = \begin{pmatrix} c_{0y} & e^{ip_y}c_y & 0 & 0 \\ -e^{-ip_y}c_y^* & c_{0y} & 0 & 0 \\ 0 & 0 & c_{0x} & e^{ip_x}c_x \\ 0 & 0 & -e^{-ip_x}c_x^* & c_{0x} \end{pmatrix} \tag{9}$$

with

$$\begin{aligned} c_{0y} &= -\int_{-\pi}^{\pi}\frac{dp_x dp_y}{2\pi}G'^{11}(\boldsymbol{p}) \\ c_y &= \int_{-\pi}^{\pi}\frac{dp_x dp_y}{2\pi}e^{ip_y}G'^{21}(\boldsymbol{p}) \\ c_x &= \int_{-\pi}^{\pi}\frac{dp_x dp_y}{2\pi}e^{ip_x}G'_{43}(\boldsymbol{p}). \end{aligned} \tag{10}$$

Now for x/y-symmetric disorder, $c_{0x} = c_{0y}$ and the kernel of the Hamiltonian of the pure system is [1]

$$H_0(t_0,\boldsymbol{p}) = \frac{1}{2}\begin{pmatrix} 0 & a & 1 & 1 \\ -a^* & 0 & -1 & 1 \\ -1 & 1 & 0 & b \\ -1 & -1 & -b^* & 0 \end{pmatrix} \tag{11}$$

with $a = 1 - t_0e^{ip_y}$ and $b = 1 - t_0e^{ip_x}$; $t_0 = \tanh(1/k_BT)$. Therefore, the most general ansatz compatible with the self-consistency equation is

$$H'(t,\boldsymbol{p}) = H_0(t,\boldsymbol{p}) + \tfrac{1}{2}\eta\mathbb{1} \tag{12}$$

where $\mathbb{1}$ is the identity matrix. A non-vanishing $\eta \in R$ prevents H' from getting eigenvalues $= 0$ for any value of p so that the divergency of the specific heat in the pure model vanishes. In a large-scale approximation, one recovers just the 'externally regularized' model of [3], where the possibility of the η-term was assumed for different reasons. In fact, a saddle-point calculation led to an $\eta \neq 0$ in a narrow region around the (shifted) critical temperature which corresponds to $t = t_c = \sqrt{2} - 1$.

In the present consideration of the full lattice theory, the η-term is connected with the diagonal propagators of figure 2. These vanish if $\eta = 0$, because the pure Green function $G^0_{ii}(\boldsymbol{r}, \boldsymbol{r}) = 0$. This is clear since we started from a *Majorana* field theory with real fermions. The η-term is forbidden within such a field theory. However, averaging over disorder gives non-zero contributions to the diagonal propagators of second order in the coupling g. However, as in [3], the self-consistency condition allows for $\eta \neq 0$ only in a neighbourhood of the critical temperature. Eventually the model must correspond to an effectice *Dirac* field theory in this region which allows for the diagonal entries in H'. The breaking of the discrete symmetry in [3] can be understood as 'spontaneous charge generation' in going from real to complex fermions.

In our approach the condition for a non-vanishing η deduced from (8) is

$$0 = \eta\left(1 - g\int \mathrm{d}^2 p \frac{\eta^2 + 2 + bb^*}{\det(\eta, t)}\right) \tag{13}$$

with

$$\begin{aligned} \det(\eta, t) &= \det(2H'(\eta, t)) \\ &= \eta^4 + 4\eta^2 + \eta^2(|a|^2 + |b|^2) - 4\,\mathrm{Re}(a)\,\mathrm{Re}(b) + |a|^2|b|^2 + 4. \end{aligned} \tag{14}$$

This expression is minimal for $t = t_c$; hence, there exists a region around the point $t = t_c$ determined by

$$1 = g\int \mathrm{d}^2 p \frac{\eta^2 + bb^* + 2}{\det(\eta, t)} \tag{15}$$

where $\eta \neq 0$. This equation, together with

$$t_0 = t + g\int \mathrm{d}^2 p \frac{\mathrm{e}^{\mathrm{i}p_y}[a^*(|b|^2 + \eta^2) - 2\,\mathrm{Re}(b)]}{\det(\eta, t)} \tag{16}$$

allows one to calculate $\eta(t_0, g)$ and $t(t_0, g)$ in this region. The phase boundaries to the 'outer phase' with $\eta = 0$ are given by

$$1 = g\int \mathrm{d}^2 p \frac{bb^* + 2}{\det(\eta = 0, t(t_0^{\pm}, g))}. \tag{17}$$

For $\eta = 0$, equation (16) leads to

$$t_0 = t + g\left[\frac{t}{1 + t^2} + \frac{3t^2 - 1}{4t(1 - t^2)}\right] F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right) - g\frac{3t^2 - 1}{4t(1 - t^2)} \tag{18}$$

where $F(k^2)$ is the hypergeometric function and $k = 4t(1 - t^2)/(1 + t^2)^2$. Figure 3 shows the effective t as a function of t_0 and the two values t_- and t_+. In the next section it is shown that the spontaneous magnetization vanishes in the new phase characterized by $\eta \neq 0$. Therefore, t_+ has to be viewed as the transition point from the ferromagnetic to the disordered phase. As expected, this leads to a lowered critical temperature.

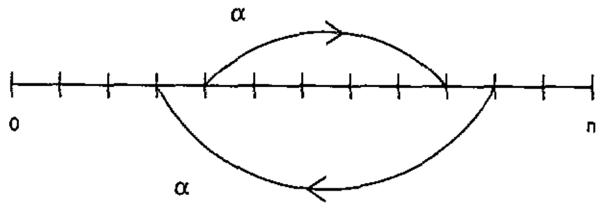

Figure 3. A two-propagator term in the expansion of $\langle \exp \frac{1}{2}\xi Q\xi \rangle_{Z_D}$. The horizontal line indicates the section of the x-axis between the two spins. The distance between them in units of the lattice spacing is n.

3. The spontaneous magnetization in the disordered phase

To get the spontaneous magnetization, we calculate the averaged spin–spin correlation function along the x axis, $\overline{S(n)} = \overline{\langle \sigma(0)\sigma(n) \rangle}$, for large values of the spin distance n measured in units of the lattice spacing. This quantity is related to the magnetization through the expression [1]

$$\overline{\mathcal{M}^2} = \lim_{n\to\infty} \overline{S(n)}. \tag{19}$$

The discussion that follows has two limitations:

(i) We use the effective Hamiltonian derived in section 1, namely, the inverse of the disorder averaged Green function for the fermion operators. This leads to the omission of certain contributions to the averaged correlation function of the spin operators which are non-local in the fermions [1]. Nevertheless, this is correct within the $N = \infty$ approximation because the omitted terms are of higher orders in $1/N$. For a specific configuration D of disorder, $S_D(n)$ can be written as a ratio of two partition functions [1]

$$S_D(n) = \prod_{i=0}^{n-1} t_x(i,0) \frac{Z'_D}{Z_D} \tag{20}$$

where in Z'_D the bonds along the x-axis between the sites $(0,0)$ and $(n,0)$ are modified from $t_x(i,0)$ to $t_x^{-1}(i,0)$. This amounts to replacing $\xi H_D \xi$ in the functional integral by $\xi H_D \xi + \frac{1}{2}\xi Q \xi$ and

$$\tfrac{1}{2}\xi Q\xi = \frac{1 - t_x^2(i,0)}{t_x(i,0)} \sum_{i=0}^{n-1} \xi_3(i,0)\xi_4(i+1,0). \tag{21}$$

Now $\langle \exp \frac{1}{2}\xi Q\xi \rangle_{Z_D}$ is expanded (see [4]) in a series of closed loops of products of fermionic propagators attached to the line between $(0,0)$ and $(n,0)$ (see figure 4). Averaging these quantities leads to contributions of the form depicted in figures 5, 6 and 7. If we replace $t_x(i,0)$ in formulae (20) and (21) by t and H_D by H' as derived in section 1, the graphs of figure 5 and 6 are counted but those of figure 7 are not. However, only the first two are of order N^0 in the $1/N$ expansion. This corresponds to the fact that the level of renormalization-group treatment is not reached in this consideration of 'mean-field level', which is suitable to handle the non-trivial phase diagram of the problem. Nevertheless, the graphs of figure 7 can be factorized out of the contributions which are accounted for in the expansion of the averaged correlation function. Therefore, they cannot affect the

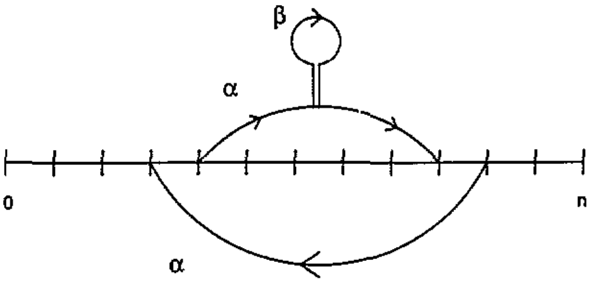

Figure 4. A term in the average of the expression $\langle \exp \frac{1}{2}\xi Q\xi \rangle_{Z_D}$, taken into account using the effective G' of section 1. α is a fixed replica index and β runs from 1 to N.

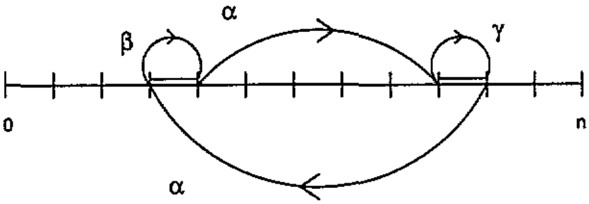

Figure 5. A term taken into account by replacing t_0 by t. β and γ run from 1 to N.

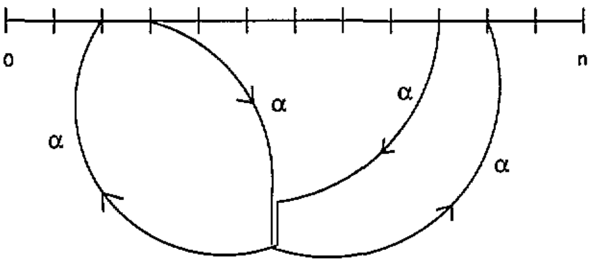

Figure 6. A term which is omitted in the $N \to \infty$ approximation.

result of the calculation that follows, i.e. they cannot lead to a non-vanishing spontaneous magnetization in the new phase.

(ii) The results are only valid in a small region around t_+, i.e. for small η and well defined $G_0(t(t_0, g))$. In addition, we use Szegö's lemma [1] to evaluate the spin-correlation function of the associated 'pure' system, i.e. a system with modified coupling t instead of t_0. This gives results only in the $n \to \infty$ limit. Nevertheless, our result is applicable for large but finite n, because in the vicinity of t_+ the strong n-dependence of $\overline{S}(n)$ in the $\eta \neq 0$ region is multiplied by an almost constant factor coming from the non-vanishing spontaneous magnetization for $t < t_c$. Moreover, the width of the region of validity depends only on t_+, not on n. The approximations made in the appendix only affect the analytical form of the dependence of the correlation lenght on η (see equation (37)). They have no influence on the qualitative behaviour of the averaged correlation function. Close to t_+, our result is correct at least in the sense of the 'scaling-limit' results obtained in the pure model

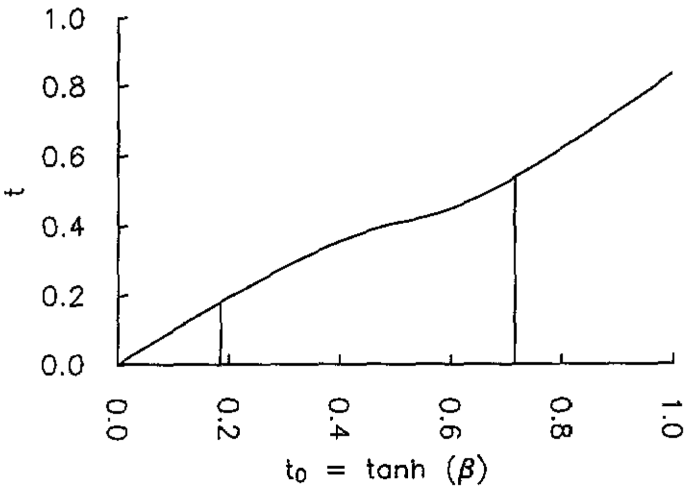

Figure 7. The effective bond strength t as a function of t_0. Vertical lines denote $t_- = 0.18$ and $t_+ = 0.71$, respectively, the new phase lying between them. The value of g is 0.3.

[1].

Following [1], we write for the square of the disorder-averaged spin correlation function

$$\overline{S(n)}^{\,2} = t^{2n} \det\left(\mathbf{1} - \frac{(1-t^2)}{t} G'(t,\eta) Q'\right). \tag{22}$$

Q' projects on the (3,4)- (i.e. horizontal) sector of the four fermions and is off-diagonal in the position space index,

$$Q'_{i,j}(x_1, y_1; x_2, y_2) = \delta_{i,3}\delta_{j,4}\delta_{x_2,x_1+1}\chi(x_1)\delta_{y_1,0}\delta_{y_2,0} - \delta_{i,4}\delta_{j,3}\delta_{x_1,x_2+1}\chi(x_2)\delta_{y_1,0}\delta_{y_2,0} \tag{23}$$

and $\chi(x)$ is the characteristic function on the interval $[0, n]$. Eventually, Q' projects G' on a $2(n+1) \times 2(n+1)$-dimensional subspace. Therefore, we have to evaluate the determinant of the $2(n+1) \times 2(n+1)$ matrix

$$\widetilde{t\mathbf{1}} + (1-t)G'Q' \tag{24}$$

where $\widetilde{t\mathbf{1}}$ means the $2(n+1) \times 2(n+1)$ matrix

$$\begin{pmatrix} t & & & & & & \\ & \ddots & & & & 0 & \\ & & t & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & t & \\ & 0 & & & & & \ddots & \\ & & & & & & & t \end{pmatrix}. \tag{25}$$

Now $G' = (H_0(t) + \eta\mathbf{1})^{-1}$ is expanded to second order in η, $(G_0 = H_0(t)^{-1})$,

$$G'(t,\eta) \simeq G_0(t)(\mathbf{1} - \eta G_0(t) + \eta^2 G_0^2(t)). \tag{26}$$

Therefore, the determinant of (24) becomes

$$\det(C + (1-t)G_0(-\eta G_0 + \eta^2 G_0^2)Q') \tag{27}$$

with

$$C = \tilde{t}\mathbf{1} + (1-t)G_0 Q'. \tag{28}$$

Eventually, the determinant factorizes to

$$\overline{S(n)}^{\,2} = \det C \det(\mathbf{1} + (1-t)C^{-1}G_0(-\eta G_0 + \eta^2 G_0^2)Q'). \tag{29}$$

The matrix C gives the averaged correlation function if η is absent. Because we are in the region where the effective coupling t is greater than t_c, $\det C$ reaches a constant non-zero value in the limit $n \to \infty$:

$$\lim_{n\to\infty} \det C = [1 - (\sinh 2\beta)^{-4}]^{1/2} \tag{30}$$

$\beta = \tanh^{-1}(t)$. Equation (30) gives the averaged square of the magnetization in the 'outer' phase, i.e. for $t < t_-$ or $t > t_+$. It leads to a non-vanishing magnetization for $t > t_+$ as in the pure system, although the functional dependence of the effective t on t_0 has to be taken into account. However,in the framework of the $N \to \infty$ limit the averaged magnetization exhibits a finite jump at the point t_+. This means that the phase transition from the magnetically ordered to the disordered state occurs at a temperature corresponding to t_+. The spontaneous magnetization vanishes for $t < t_+$, whereas it differs from zero for $t > t_+$. To see this, we calculate the second determinant in (29) for $0 < \eta \ll 1$.

The block structure of C is [1]

$$\begin{pmatrix} & & & 0 & & & & \\ & A & & \vdots & & & 0 & \\ & & & 0 & & & & \\ b_0 & \dots & b_{n-1} & 1 & & & & \\ & & & & 1 & b_{n-1}^* & \dots & b_0^* \\ & & & & 0 & & & \\ & 0 & & & \vdots & & A^\dagger & \\ & & & & 0 & & & \end{pmatrix} \tag{31}$$

where A is an $n \times n$ Toeplitz matrix with the elements

$$A_{j,k} = \int_{-\pi}^{\pi} \frac{\mathrm{d}p}{2\pi} f(p) \mathrm{e}^{\mathrm{i}p(j-k)} \tag{32}$$

where we choose the indices j, k to run from $-(n-1)/2$ to $(n-1)/2$, and

$$b_j = \int_{-\pi}^{\pi} \frac{\mathrm{d}p}{2\pi} f(p) \mathrm{e}^{\mathrm{i}p(n-j)} \tag{33}$$

for $j \in [0, n-1]$. $f(p)$ is a unimodular function parametrized by t and its dual $\tilde{t}$, $\tilde{t} = (1-t)/(1+t)$:

$$f(p) = \left(\frac{(t - \tilde{t}\mathrm{e}^{\mathrm{i}p})(t\tilde{t} - \mathrm{e}^{\mathrm{i}p})}{(t\mathrm{e}^{\mathrm{i}p} - \tilde{t})(t\tilde{t}\mathrm{e}^{\mathrm{i}p} - 1)} \right)^{1/2}. \tag{34}$$

The second factor in equation (28) can now be written in the form

$$\exp(\mathrm{Tr}\log(\mathbf{1} + (1-t)C^{-1}G_0(-\eta G_0 + \eta^2 G_0^2)Q')). \tag{35}$$

The logarithm can be expanded in powers of η, because we are not in the critical regime and therefore $G_0(t)$ is a bounded operator in its domain of definition and has finite operator norm. By an appropriate choice of η (close enough to zero), the expansion is well defined. In the first approximation, terms higher than quadratic are neglected. It is shown in the appendix that the term

$$\mathrm{Tr}(1-t)C^{-1}G_0^2 Q' \tag{36}$$

which is proportional to η vanishes, and the term proportional to η^2 is negative definite, yielding finally for large n

$$\overline{S(n)}^{\,2} \simeq \text{constant} \times \exp[-(\eta^2\gamma(t))n] \tag{37}$$

where $\gamma(t)$ is a positive constant depending on t_0 and the disorder strenght g. Clearly, the disorder-averaged spin correlation function decays exponentially for an arbitrary non-zero value of η. Eventually, the averaged spontaneous magnetization vanishes in the phase with $\eta \neq 0$.

4. Conclusions

In this paper, the phase transition from the ferromagnetic ordered phase to the disorder-dominated phase in the two-dimensional random bond Ising model is investigated. Whereas in previous approaches [2–4] a large-scale approximation in the vicinity of an assumed critical region allowed to reduce the number of independent fermion degrees of freedom from four to two, we treat the full lattice theory in a self-consistent way in the framework of a $1/N$ expansion. This expansion is not supposed to correspond to any special physical properties of the system but serves as a bookkeeping device, which allows one to extract the most important 'mean field' contributions usually omitted in renormalization-group approaches (i.e. tadpoles). The most general self-consistent ansatz for the effective Hamiltonian has the same structure as the saddle-point solution of the Q matrix theory [3], although the reason for introducing a possibly non-vanishing η is different: in the Q-matrix theory one uses supersymmetric fields and the η-term regularizes the integration over the bosons. After averaging over the disorder, η becomes non-zero in a certain region due to spontaneous symmetry breaking. The order parameter in the symmetry-broken phase can therefore be identified with η, leading to a continous phase transition from the ferromagnetic to the disorder-dominated phase as well as from this phase to the paramagnetic one. The question arises whether the spontanous magnetization in the new phase vanishes or not. To this end, we calculated the asymptotic behaviour (i.e. $n \to \infty$) of the disorder-averaged spin correlation function in a region around the transition line from the magnetic ordered phase to the new phase. Whereas the magnetization on the ferromagnetic side is finite, corresponding to $T < T_c$, the correlation function decays exponentially

$$\overline{S(n)} \simeq \exp -n/\xi$$

with

$$\xi = \frac{2}{\gamma(t)\eta^2}.$$

This means that η^2 plays the role of inverse correlation length. However, it is not clear what kind of operators become massless as ξ goes to infinity. The Q-matrix analysis shows a massless mode of the local composite Q operators at t_+. Eventually, a rigorous RG treatment of the transition point has to deal with these objects. The temperature dependence of the magnetization in the ferromagnetic phase can be calculated by means of equations (18) and (30). The appearance of η^2 and not η in the expression for the averaged correlation function is due to the fact that the self-consistent equation allows two real solutions with η positive or negative, whereas the physics must not depend on the sign of η. This corresponds to the freedom of choosing the sign of the regularization term in the supersymmetric theory. If one relates the averaged asymptotic correlation function to the averaged square of the spontaneous magnetization in the usual way, it follows that the new phase shows no long-range magnetic order. Therefore, the only possibility to distinguish it from the paramagnetic region is to investigate the relaxational dynamics for the spins; this will be discussed elsewhere. However, the present paper says nothing about the transition from the new phase to the paramagnetic one. It is even possible that something happens in the new phase when the effective t reaches t_c. In the framework of the $N \to \infty$ saddle-point calculation, the spontaneous magnetization exhibits a finite jump at the transition from the ordered to the new phase. Whether this remains correct after one takes fluctuations into account, can perhaps be answered via renormalization group treatment of the theory on the ferromagnetic side.

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Appendix

To calculate the term proportional to η in equation (36), we write $G_0(p)$ as

$$G_0 = \frac{1}{\det H_0} \begin{pmatrix} \alpha_1 & \beta_1 & \epsilon & \delta \\ -\beta_1^* & \alpha_1^* & -\delta^* & \epsilon^* \\ -\epsilon^* & \delta & \alpha_2 & \beta_2 \\ -\delta^* & -\epsilon & -\beta_2^* & \alpha_2^* \end{pmatrix} \tag{A1}$$

with

$$\begin{aligned} \alpha_1 &= b - b^* \\ \alpha_2 &= a^* - a \\ \beta_1 &= (b + b^*) - abb^* \\ \beta_2 &= (a + a^*) - baa^* \\ \epsilon &= ab^* - 2 \\ \delta &= ab - 2 \end{aligned} \tag{A2}$$

and the conventions of equation (11). The (3,4)- (i.e. horizontal) sector of $G_0^2(p)$ then reads

$$\frac{1}{\det H_0^2}\begin{pmatrix} -(|\alpha_2|^2+|\beta_2|^2+|\epsilon|^2+|\delta|^2) & 0 \\ 0 & -(|\alpha_2|^2+|\beta_2|^2+|\epsilon|^2+|\delta|^2) \end{pmatrix}. \tag{A3}$$

Therefore, $G_0^2 Q'$ has the form

$$\begin{pmatrix} \mathbf{0} & * \\ * & \mathbf{0} \end{pmatrix} \tag{A4}$$

where $\mathbf{0}$ and $*$ are $(n+1)\times(n+1)$ matrices. Multiplication from the left with C^{-1} yields a matrix with vanishing diagonal entries. It follows that the contribution proportional to η vanishes.

To calculate the quadratic term, we write

$$\mathrm{Tr}(C^{-1}G_0^3 Q') = \mathrm{Tr}(G_0^2(C-\tilde{t}\mathbf{1})C^{-1}) = \widetilde{\mathrm{Tr}}(G_0^2) - \widetilde{\mathrm{Tr}}(G_0^2\tilde{t}\mathbf{1}C^{-1}) \tag{A5}$$

and $\widetilde{\mathrm{Tr}}$ means the trace in the $2(n+1)\times 2(n+1)$-dimensional subspace of the horizontal sector under consideration. To give an estimate for the matrix C^{-1}, we use the fact that the matrix A in equation (31) is 'almost unitary', in the sense that

$$AA^\dagger = \mathbf{1} + \frac{1}{n}B \tag{A6}$$

where B has bounded matrix elements. This is due to the fact that

$$\sum_{j=-(n-1)/2}^{(n-1)/2} g(p)\mathrm{e}^{\mathrm{i}pj} = g(0) + \mathrm{O}(1/n) \tag{A7}$$

for a bounded function $g(p)$, and $f(p)$ in equation (32) is of unit modulus. Then it is possible to write for C^{-1}

$$C^{-1} = \begin{pmatrix} & A^\dagger & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & & 0 & \\ d_0 & \cdots & d_{n-1} & 1 & & & \\ & & & & 1 & d_{n-1}^* & \cdots & d_0^* \\ & 0 & & & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & & A & \end{pmatrix} + \frac{1}{n}B' \tag{A8}$$

with

$$d_j = -\sum_{i=0}^{n-1} b_i A^\dagger_{i-(n-1)/2,\, j-(n-1)/2} = \mathrm{O}(1/n). \tag{A9}$$

With the definition

$$G_{0,jk}^2 = G_0^2(x_1=j, y_1=0; x_2=k, y_2=0) = \int_{-\pi}^{\pi}\frac{\mathrm{d}p}{2\pi}h(p)\mathrm{e}^{\mathrm{i}p(j-k)} \tag{A10}$$

for $j, k \in [0, n]$, we get

$$\widetilde{\mathrm{Tr}}(G_0^2\tilde{t}\tilde{1}C^{-1}) = 2G_{0,nn}^2 + 2nti \int_{-\pi}^{\pi} \frac{\mathrm{d}p}{2\pi} h(p)\,\mathrm{Im}(f(p)) + \mathrm{O}(1). \tag{A11}$$

The contribution from the integral leads to at most an oscillating factor for the averaged spin-correlation function, but actually it vanishes due to the symmetry properties of $h(p)$ and $f(p)$. It follows that the relevant contribution is the first term in equation (A5), namely

$$\widetilde{\mathrm{Tr}} G_0^2 = 2(n+1)G_{0,nn}. \tag{A12}$$

Finally, one gets for the (positive definite) $\gamma(t)$ of equation (37)

$$\gamma(t) = 2(1-t)\int_{-\pi}^{\pi} \frac{\mathrm{d}p_x \mathrm{d}p_y}{(2\pi)^2} \frac{|\alpha_2|^2 + |\beta_2|^2 + |\epsilon|^2 + |\delta|^2}{\det H_0(t)^2}. \tag{A13}$$

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